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A Variational Technique for Smoothing Flight-Test and Accident Data

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The problem of determining aircraft motions along a trajectory is solved using a variational algorithm that generates unmeasured states and forcing functions, and estimates instrument bias and scale-factor errors. The problem is formulated as a nonlinear fixed-interval smoothing problem, and is solved as a sequence of linear two-point boundary value problems, using a sweep method. The algorithm has been implemented for use in flight-test and accident analysis. Aircraft motions are assumed to be governed by a six-degree-of-freedom kinematic model; forcing functions consist of body accelerations and winds, and the measurement model includes aerodynamic and radar data. Examples of the determination of aircraft motions from typical flight-test and accident data are presented.

Introduction

ACCURATE determination of aircraft motions along a flight trajectory, with estimation of instrument bias and scale-factor errors, is an important problem in the analysis of flight-test measurements. A similar, more difficult problem occurs in accident analysis, where the desired aircraft motions may have to be determined from an en route radar track or from a foil flight record. Each of these problems can be formulated as a nonlinear, fixed-interval smoothing problem. This paper describes a variational technique for solving such a problem, with application to trajectory reconstruction and parameter estimation in flight-test and accident analysis.

It should be emphasized that algorithms for postflight data processing abound[1-5]—most utilize a version of an iterated, extended Kalman filter (IEKF). Although good results have been reported using the IEKF in off-line data processing applications, that algorithm is not optimum in its use of future as well as past data in the measurement record. The few techniques for aircraft accident analysis found in the open literature[6,7] are primarily algebraic; the desired solutions are obtained in terms of the available measurements and their time derivatives. These are "optimum" only in the sense that individual data records are smoothed and the required derivatives obtained.

The technique advocated here for flight-test and accident analysis is based on a variational solution of the nonlinear, fixed-interval smoothing problem. It is iterative in nature, providing improved state and forcing-function estimates until a minimum performance measure is achieved. Linearization is about a smoothed trajectory and convergence is quadratic. Its analytical basis is not new—it is an example of the "successive sweep" procedure of McReynolds and Bryson,[8] originally devised to solve a continuous optimal control problem. Although the algorithmic development presented here extends and unifies previous work, the main contribution of this paper is in applying the variational procedure to solve the general problem of determining aircraft motions along a flight trajectory.

Outlined here are two equivalent algorithms for solution of the discrete smoothing problem, details of which have been published previously.[9] The first algorithm, which consists of a forward covariance filter and backward smoother, facilitates comparison of the variational and extended Kalman filter methods. The second algorithm, which consists of a backward information filter and forward smoother, is shown to have certain computational advantages. It has recently been implemented at Ames Research Center for use in flight-test analyses and to assist the National Transportation Safety Board in its investigation of aircraft accidents. Included here are a discussion of the dynamic and measurement models used in the program, and examples illustrating application of the program in analyzing the consistency of flight-test data and aircraft motions from radar data.

Smoothing Algorithms

The fixed-interval smoothing problem is defined as follows. Given a system with state model

$$x(i+1)=f[x(i),u(i)] \qquad x(0)=x_0 \tag{1}$$

and measurement model

$$a(i)=u(i)-w(i) \qquad z(i)=h[x(i)]+v(i) \tag{2}$$

determine x_0 and the sequence $[w(i)]$, $i=0,\ldots,N-1$ that minimize the performance measure

$$J=\tfrac{1}{2}(x_0-\bar{x}_0)^T P_0^{-1}(x_0-\bar{x}_0)$$

$$+\tfrac{1}{2}\sum_{i=0}^{N-1}[w^T(i)Q^{-1}w(i)+v^T(i+1)R^{-1}v(i+1)] \tag{3}$$

This formulation of the smoothing problem suitably models flight-test data format and the numerical techniques used with digital computers. In Eq. (2) the vector $w(i)$ is the error in measuring the system input $u(i)$, and $v(i)$ is the error in measuring the system output $h[x(i)]$. Note that when any element of $a(i)$ is unavailable, the corresponding element of $w(i)$ is considered to be an unknown forcing function. In Eq. (3), $\bar{x}_0$ is an a priori estimate of the state at the initial time, and P_0, Q, and R are weighting matrices. Sage and Melsa[10] discuss a Bayesian maximum likelihood interpretation of the performance measure in which P_0 is the error covariance matrix for the a priori estimate, and Q and R are error covariance matrices for the input and output measurement sequences, respectively (assumed stationary).

The nonlinear smoothing problem is solved using a method of successive approximations based on expansion of the

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performance measure [Eq. (3)] to the second order and of the dynamic constraint [Eq. (1)] to the first order. Suppose we choose an x_0, $[w(i)]$ and obtain a nominal trajectory by solving Eq. (1). It is unlikely that our solution minimizes J, but we shall try to determine a neighboring solution that yields a smaller value. To do this, we first express a variation δJ in the performance measure in terms of variations δx_0 and $[\delta w(i)]$:

$$\delta J = (x_0 - \bar{x}_0)^T P_0^{-1}\delta x_0 + \tfrac{1}{2}\delta x_0^T P_0^{-1}\delta x_0 + \sum_{i=0}^{N-1}[w^T(i)Q^{-1}\delta w(i) + \tfrac{1}{2}\delta w^T(i)Q^{-1}\delta w(i)] + \sum_{i=0}^{N-1}[-v^T(i+1)R^{-1}h_x\delta x(i+1) + \tfrac{1}{2}\delta x^T(i+1)h_x^T R^{-1}h_x\delta x(i+1)] \tag{4}$$

Next, we assume that deviation from the nominal trajectory will be governed by

$$\delta x(i+1) = f_x\delta x(i) + f_w\delta w(i), \delta x_0 \tag{5}$$

In Eqs. (4) and (5), the partial derivatives are defined as

$$f_x = \frac{\partial f[x(i),u(i)]}{\partial x(i)} \qquad f_w = \frac{\partial f[x(i),u(i)]}{\partial w(i)}$$

$$h_x = \frac{\partial h[x(i+1)]}{\partial x(i+1)} \tag{6}$$

and are to be evaluated along the nominal trajectory.

Our objective now is to specify δx_0, $[\delta w(i)]$ such that δJ is as large a negative number as possible, subject to the dynamic constraint of Eq. (5). We solve this "accessory minimization" problem in the usual way[11] by adjoining the constraint to Eq. (4) using a Lagrange multiplier:

$$\delta \bar{J} = \delta J + \sum_{i=0}^{N-1}\lambda^T(i+1)[f_x\delta x(i) + f_w\delta w(i) - \delta x(i+1)] \tag{7}$$

The necessary conditions for minimizing $\delta\bar{J}$ lead to a linear, two-point boundary value problem (LTPBVP) given by Eq. (5) and

$$\lambda(i) = f_x^T[\lambda(i+1) + h_x^T R^{-1}h_x\delta x(i+1) - h_x^T R^{-1}v(i+1)]$$

$$\lambda(0) = -P_0^{-1}[(x_0 - \bar{x}_0) + \delta x_0] \qquad \lambda(N) = 0 \tag{8}$$

$$\delta w(i) = -w(i) - Qf_w^T f_x^{-T}\lambda(i) \tag{9}$$

This LTPBVP has an exact solution, as is well known. Hence it is possible to determine δx_0, $[\delta w(i)]$, recompute the nominal trajectory with

$$x_0 \leftarrow x_0 + \delta x_0 \qquad w(i) \leftarrow w(i) + \delta w(i)$$

and evaluate the performance measure, iterating until J is minimized. The change in J that should be realized at any iteration is found by substituting Eqs. (8) and (9) into Eq. (7):

$$\delta J = -\tfrac{1}{2}\delta x_0^T P_0^{-1}\delta x_0 - \tfrac{1}{2}\sum_{i=0}^{N-1}[\delta w^T(i)Q^{-1}\delta w(i) + \delta x^T(i+1)h_x^T R^{-1}h_x\delta x(i+1)] \tag{10}$$

Two equivalent sweep solutions of the LTPBVP are given here. The first is derived by introducing a vector $\delta\hat{x}(i)$ and matrix $P(i)$ and letting

$$\delta x(i) = \delta\hat{x}(i) - P(i)\lambda(i) \tag{11}$$

Notice that the boundary conditions of Eq. (8) require that

$$\delta\hat{x}(0) = \bar{x}_0 - x_0 \qquad P(0) = P_0 \qquad \delta x(N) = \delta\hat{x}(N)$$

Straightforward algebraic manipulation yields the algorithm outlined in Appendix A, which is essentially the procedure proposed by Cox[12] in 1965. It consists of a forward covariance filter and a backward smoother. A disadvantage is that it requires a matrix P_0 that specifies uncertainty in the a priori estimate $\bar{x}_0$. The algorithm is in a form suitable for comparison with the usual extended Kalman filter often employed for nonlinear state and parameter estimation.[13]

We observe that, for a class of systems with the state model [Eq. (1)] in the form

$$f[x(i),u(i)] = g[x(i)] + \Gamma u(i)$$

the forward covariance filter of Appendix A is identical to an extended Kalman filter linearized about a (prior) nominal solution. The usual linearization, however, is about a current solution. In at least one case,[5] the extended Kalman filter can be coupled with a backward smoother. Such a procedure requires no starting solution but does not iterate to minimize a performance measure, and so provides only an approximate solution of the nonlinear smoothing problem. A numerical comparison of algorithms is part of an ongoing study. This author is not aware of any comparison of nonlinear smoothing techniques like that made of filtering techniques by Wishner et al.[14]

Another, more useful sweep solution of the LTPBVP is obtained by introducing a vector $\alpha(i)$ and matrix $M(i)$ and letting

$$\lambda(i) = \alpha(i) + M(i)\delta x(i) \tag{12}$$

In this case the boundary conditions of Eq. (8) require that $\alpha(N) = 0$, $M(N) = 0$, and

$$\delta x_0 = -[P_0^{-1} + M(0)]^{-1}[P_0^{-1}(x_0 - \bar{x}_0) + \alpha(0)] \tag{13}$$

The resulting algorithm, outlined in Appendix B, consists of a backward information filter and a forward smoother. An advantage of this formulation is that the covariance of the a priori estimate need not be considered [P_0^{-1} set to zero in Eq. (13)]. Notice that the sequences $[d(i)]$, $[L(i)]$ computed during the filter pass, are utilized during the smoothing pass. Here the storage requirement depends on the dimensions of $w(i)$ and $x(i)$ and, of course, on the length of the data record.

The analyst must be careful in choosing starting values for x_0 and $[w(i)]$, and in selecting the weighting matrices P_0, Q, and R. The convergence properties of the algorithm are influenced directly by the nominal solution generated by the initial choice of x_0 and any unmeasured forcing functions. Our experience with this problem indicates that suitable results are obtained by solving the finite-difference approximation of the state model for the forcing-function sequences, using filtered versions of the measurement records. On the other hand, the nature of the solution, once convergence is obtained, depends to a considerable degree on the choice of the weighting matrices P_0, Q, and R. The effect of P_0 is to bias the estimate x_0 toward the a priori value $\bar{x}_0$; it can easily be ignored when applying the algorithm of Appendix B. Reasonable values for the elements of Q and R may be determined as follows: Filter each measurement record until the residual sequence appears sufficiently "white," and use its variance as the appropriate diagonal element; for an unmeasured forcing function, use the mean-square value of the starting sequence $[w(i)]$.

We should observe in passing that, heuristically, smoothing is a process of zero-phase-shift filtering in which bandwidth increases as the scale of Q increases and the scale of R decreases. One expects that forcing function and residual variances will agree with the corresponding elements of Q and R used in obtaining the solution. That solution, however, is not unique. Scaling the elements of Q up and the elements of R down by the same factor (bandwidth increase) will result in a solution having closer fits to the data, but with "noisier" forcing-function estimates. This fact merely emphasizes the need for the analyst to carefully consider the engineering aspects of his problem.

Aircraft Motion Analysis

For the applications considered in this paper, aircraft motions are assumed to be governed by a six-degree-of-freedom kinematic model, referred to a flat, nonrotating Earth.[15] Aircraft states consist of angular velocities (p,q,r), linear velocities (u,v,w), Euler angles (ϕ,θ,ψ), and vehicle position (x,y,z). Forcing functions for the model are angular accelerations (a_l,a_m,a_n), and linear accelerations (a_x,a_y,a_z). When motion of the air mass is considered, the model is augmented with wind velocities (w_x,w_y,w_z) and accelerations (g_x,g_y,g_z). The differential equations defining a continuous state model are given by

$$\dot{p}=a_l \qquad \dot{q}=a_m \qquad \dot{r}=a_n \tag{14}$$

$$\dot{u}=a_x-qw+rv-g\sin\theta$$

$$\dot{v}=a_y-ru+pw+g\cos\theta\sin\phi \tag{15}$$

$$\dot{w}=a_z-pv+qu+g\cos\theta\cos\phi$$

$$\dot{\phi}=p+(q\sin\phi+r\cos\phi)\tan\theta$$

$$\dot{\theta}=q\cos\phi-r\sin\phi \tag{16}$$

$$\dot{\psi}=(r\cos\phi+q\sin\phi)/\cos\theta$$

$$\begin{bmatrix}\dot{x}\\ \dot{y}\\ \dot{z}\end{bmatrix}=L_{VB}\begin{bmatrix}u\\ v\\ w\end{bmatrix}\qquad\begin{bmatrix}\dot{w}_x\\ \dot{w}_y\\ \dot{w}_z\end{bmatrix}=\begin{bmatrix}g_x\\ g_y\\ g_z\end{bmatrix} \tag{17}$$

The matrix L_{VB} transforms from a body-fixed reference frame to a vehicle-carried vertical frame, and is given by

$$L_{VB}=\begin{bmatrix}\cos\theta\cos\psi & \sin\phi\sin\theta\cos\psi-\cos\phi\sin\psi & \cos\phi\sin\theta\cos\psi+\sin\phi\sin\psi\\ \cos\theta\sin\psi & \sin\phi\sin\theta\sin\psi+\cos\phi\cos\psi & \cos\phi\sin\theta\sin\psi-\sin\phi\cos\psi\\ -\sin\theta & \sin\phi\cos\theta & \cos\phi\cos\theta\end{bmatrix}$$

Elements of the measurement model are generally among the angular rates, Euler angles, and position coordinates. Also included are functions of the state variables, such as radar information (slant range R, bearing angle B, and altitude h), where

$$R=(x^2+y^2+z^2)^{1/2} \qquad B=\tan^{-1}(y/x) \qquad h=-z \tag{18}$$

and aerodynamic quantities (the airspeed, V; the angle of attack, α; and the sideslip angle, β), where

$$V=(u_a^2+v_a^2+w_a^2)^{1/2} \qquad \alpha=\tan^{-1}(w_a/u_a) \qquad \beta=\sin^{-1}(v_a/V) \tag{19}$$

and

$$\begin{bmatrix}u_a\\ v_a\\ w_a\end{bmatrix}=\begin{bmatrix}u\\ v\\ w\end{bmatrix}+L_{VB}^T\begin{bmatrix}w_x\\ w_y\\ w_z\end{bmatrix} \tag{20}$$

Note that any bias or scale-factor errors associated with the measurement model are appended to the state vector and treated as constants, that is,

$$\dot{b}=0,b_0 \qquad \dot{s}=0,s_0 \tag{21}$$

where b is a vector of unknown biases and s is a vector of unknown scale factors.

The state model described by Eqs. (14-17) is for an axis system fixed in the vehicle body. In some situations, however, it may be desirable to refer angular rates and Euler angles to an axis system that follows the flight path (a wind-axis system). This can be done simply by assuming that measurements of α and β have been made that are identically zero. The degree of approximation to a true wind-axis representation will depend on their measurement-noise covariance values, relative to those for the actual measurements.

Each nominal, or "smoothed" trajectory is computed by employing a finite-difference approximation to the continuous state model. Here we use a simple Euler procedure

$$x(i+1)=x(i)+\Delta t\dot{x}(i) \qquad x(0)=x_0 \tag{22}$$

where Δt is the time step. The computing strategy used to solve the smoothing problem is shown in Fig. 1. An iteration loop (not shown) evaluates the performance measure, exerts some cotnrol over step size δx_0, $[\delta w(i)]$, and decides when to terminate the solution. The algorithm programmed for this application is the backward filter, forward smoother outlined in Appendix B.

Two examples will be presented here. The first illustrates a postflight-test analysis of compatibility among attitude and attitude-rate measurements. The second example is a typical postaccident trajectory analysis from radar information, winds aloft, and aircraft performance data. The data for these examples were obtained during an experiment performed using the Ames CV-990 research aircraft equipped with an air-data system, inertial platform, and body-mounted accelerometers. Vehicle position was monitored by a ground-based radar station. The maneuver to be examined consists of a near-touchdown and climb-out followed by a left turn; it lasted for about 2 min and the data rate was 1 Hz.

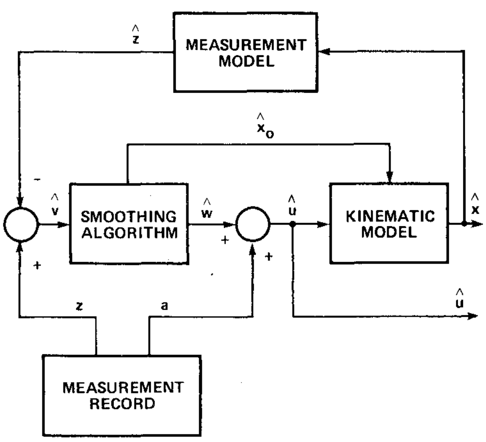

Fig. 1 Block diagram of computing strategy.

Table 1 Parameter estimates: flight-test analysis

Parameter	Value	Standard deviation
	Initial conditions	
p, rad/s	−0.3407E−01	0.1061E−01
q, rad/s	0.1469E−02	0.4315E−02
r, rad/s	−0.8669E−02	0.3802E−02
ϕ, rad	0.4840E−01	0.1704E−01
θ, rad	0.3826E−02	0.1697E−01
ψ, rad	0.1139E+00	0.4562E−02
	Process biases	
a_l, rad/s^2	0.2478E−03	0.8995E−03
a_m, rad/s^2	−0.6811E−04	0.3483E−03
a_n, rad/s^2	0.2376E−04	0.3100E−03
	Measurement biases	
p, rad/s	0.1303E−02	0.6672E−03
q, gad/s	0.1885E−02	0.3881E−03
r, rad/s	0.6531E−03	0.3228E−03
γ'rad	−0.1598E−01	0.1225E−01
ϕ'rad	0.1415E−01	0.1613E−01
	Scale factors	
p	0.1076E+01	0.2111E−01
q	0.1044E+01	0.2202E−01
r	0.1018E+01	0.1121E−01

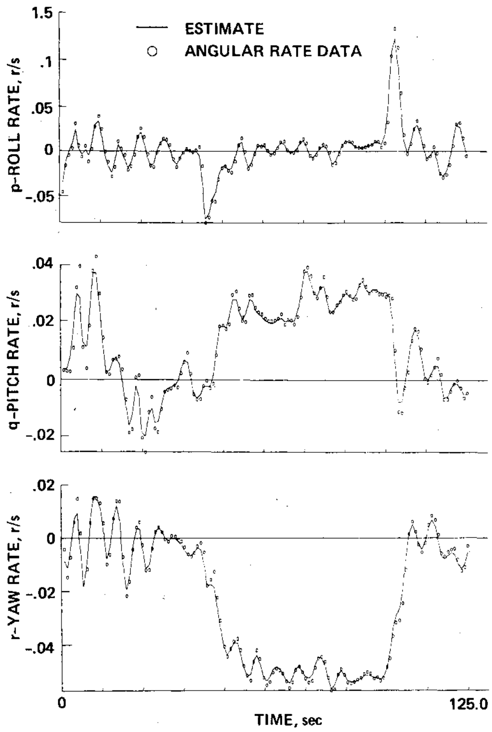

Fig. 3 Fits to angular rate data: flight-test analysis.

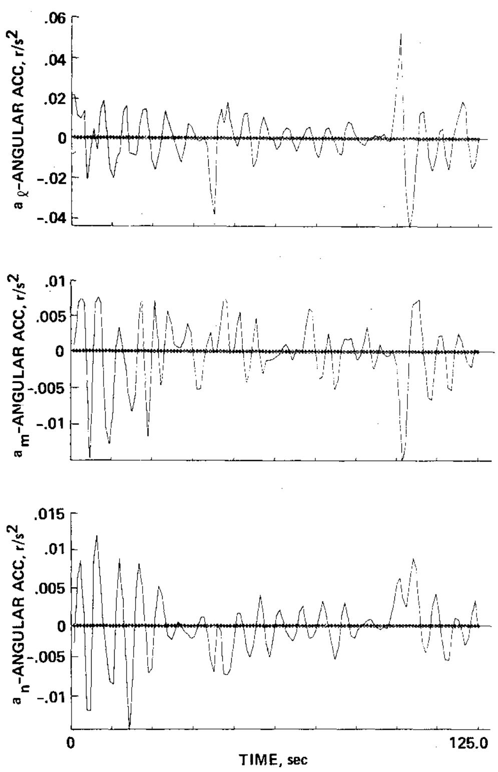

Fig. 2 Estimates of angular acceleration: flight-test analysis.

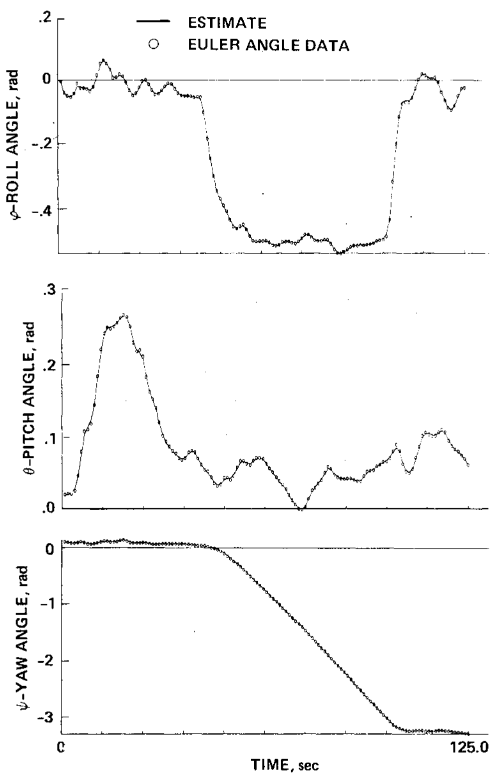

Fig. 4 Fits to Euler angle data: flight-test analysis.

Flight-Test Analysis

A simple data-consistency check following flight test involves smoothing of attitude and attitude-rate data. For this example, the state variables are p,q,r and ϕ,θ,ψ. The kinematic model is described by Eqs. (14) and (16), with angular accelerations a_l, a_m, a_n considered as unknown inputs. The data record includes measurements of all the states, and all except ψ are assumed to be biased. In addition, measurements of p,q,r are assumed to contain scale-factor errors. A solution of the smoothing problem yielding the desired initial conditions, error parameters, and forcing functions was achieved after three iterations of the algorithm. The results are shown in Table 1 and Figs. 2-4. The first column of numbers in Table 1 contains parameter estimates (initial conditions, biases, and scale factors); the second column contains estimates of their standard deviations. The latter are obtained from the diagonal terms of the information matrix inverse in Eq. (13). The term "process" bias refers to the mean value of an unknown forcing function. The time histories of Figs. 3 and 4 indicate close agreement between

Table 2 Parameter estimates: motion analysis

Parameter	Value	Standard deviation
	Initial conditions	
p, rad/s	0.1163E−01	0.1852E−01
q, rad/s	−0.8340E−02	0.5954E−02
r, rad/s	−0.2151E−02	0.5056E−02
u, m/s	0.7248E+02	0.1054E+01
v, m/s	−0.2743E+00	0.1255E+01
w, m/s	0.5396E+01	0.1116E+01
ϕ, rad	−0.6472E−01	0.3999E−01
θ, rad	0.4147E−01	0.1008E−01
ψ, rad	0.2999E−01	0.1632E−01
x,m	−0.1512E+04	0.1190E+01
y,m	−0.7346E+01	0.9230E+00
z,m	−0.3125E+02	0.1226E+01
	Process biases	
a_l, rad/s^2	−0.6301E−04	0.9956E−03
a_m, rad/s^2	0.5775E−04	0.3803E−03
a_n, rad/s^2	−0.4701E−04	0.2265E−03
a_x, m/s^2	0.9255E+00	0.8747E−01
a_z, m/s^2	−0.1027E+02	0.8877E−01

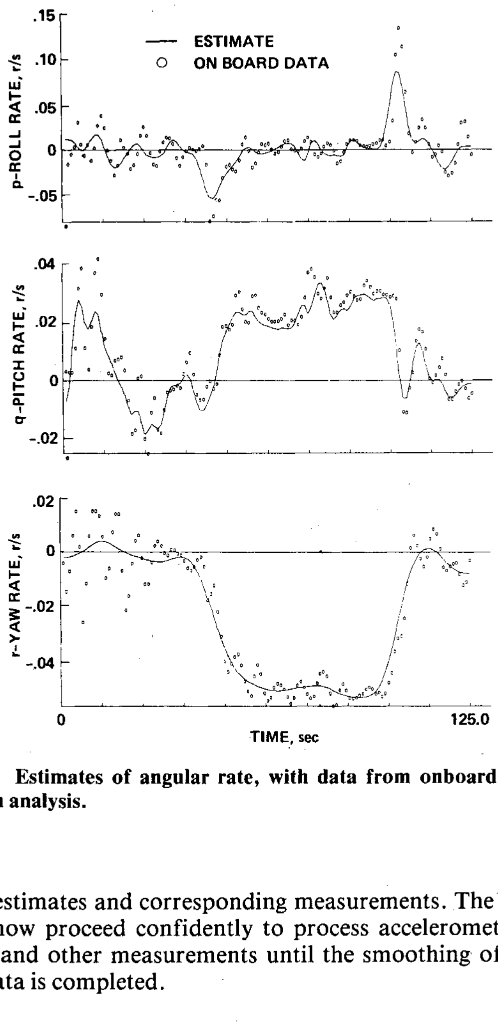

Fig. 5 Estimates of angular rate, with data from onboard system: motion analysis.

state estimates and corresponding measurements. The analyst may now proceed confidently to process accelerometer, air-data, and other measurements until the smoothing of flight-test data is completed.

Accident Analysis

This example describes a procedure for determining aircraft motions from limited data, such as that which might be available following an accident. The analysis technique utilizes data from a radar tracking system, along with local meteorological and aircraft performance data, to estimate thrust and lift forces, and Euler angles along the trajectory.

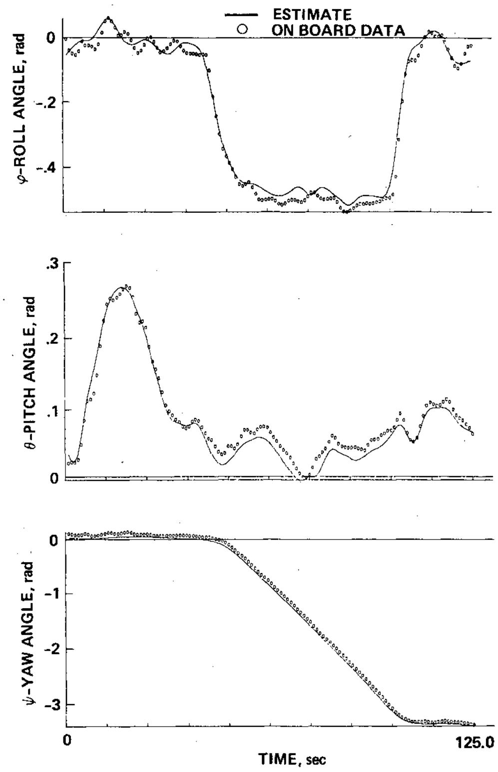

Fig. 6 Estimates of Euler angles, with data from onboard system: motion analysis.

For this problem, state variables are p,q,r; ϕ,θ,ψ; u,v,w; and x,y,z; that is, the full kinematic model [Eqs. (14-17)] is required. It is necessary to assume that vehicle side force a_y is negligible.[7] The forcing functions a_l, a_m, a_n and a_x, a_z are to be estimated. The measurement record consists of vehicle position, winds, and air temperature. We add to the record a time history $\beta=0$ (to be consistent with the assumption that $a_y=0$), and an estimate of angle of attack α. In the linear region that estimate can be approximated by

$$\alpha=\alpha_0-ma_{zw}/QSC_{L_\alpha} \qquad Q=\rho V^2/2 \tag{23}$$

where m is mass, S is wing area, ρ is air density, and C_{L_α} is the derivative of the lift coefficient with respect to α. Values of α_0 and C_{L_α} depend primarily on aircraft configuration and Mach number and are tabulated for a given aircraft. For high angles of attack, a flat-plate relationship yields a good approximation:

$$\alpha=\tan^{-1}(a_{xw}/a_{zw}) \tag{24}$$

In Eqs. (23) and (24), a_{xw}, a_{zw} are specific forces in a wind-axis system, and V is true airspeed. These quantities can be estimated by first applying the algorithm with $\alpha=0$ in the measurement record, thus effectively transforming the kinematic model to the wind-axis system. The specific forces a_{xw}, a_{zw} (excess thrust and lift) are important aircraft performance parameters[16]; together with body-axis Euler angles they are valuable aids to the investigator in visualizing an accident trajectory.

The resulting body-axis solution, shown in Table 2 and in Figs. 5-9, required six iterations for convergence. Estimates of angular rates (Fig. 5), Euler angles (Fig. 6), and body-axis specific forces (Fig. 7) are shown with data from the onboard instrumentation system for comparison. Those data were not corrected for bias or scale-factor errors. Finally, the fits to the radar record and to the "pseudoaerodynamic" data (α,β) are shown in Figs. 8 and 9. Not shown are estimates of angular accelerations (a_l,a_m,a_n), and linear velocities (u,v,w). It

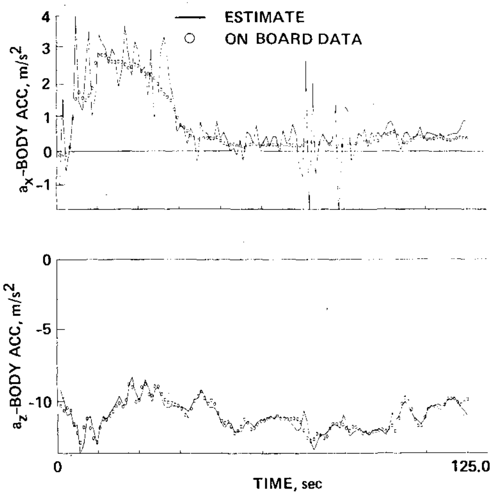

Fig. 7 Estimates of body accelerations, with data from onboard system: motion analysis.

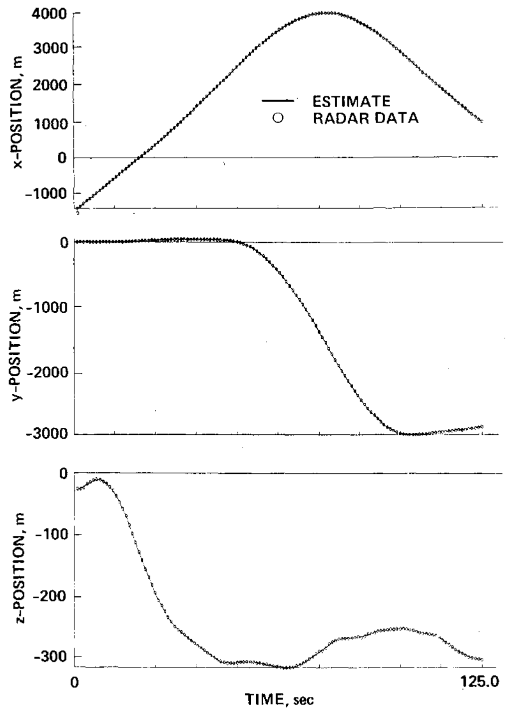

Fig. 8 Fits to radar position data: motion analysis.

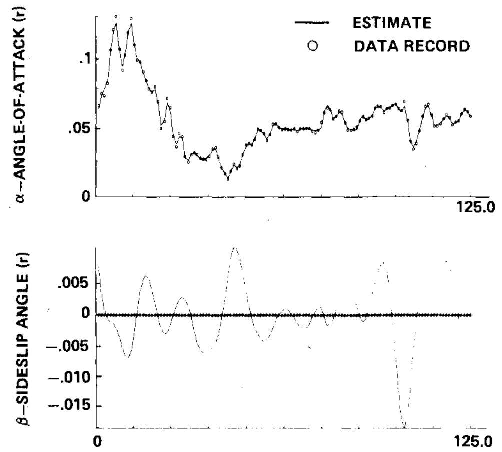

Fig. 9 Fits to aerodynamic data: motion analysis.

should be noted that the data from air traffic control (ATC) or en route radar stations are generally less frequent than one sample per second. Nevertheless, our analyses of several recent accidents have produced quite reasonable scenarios.

Concluding Remarks

The problem of determining aircraft motions along a trajectory has been considered. It has been formulated as a fixed-interval smoothing problem, and two equivalent algorithms for its solution have been presented. The solution corrects the measurements for instrument bias and scale-factor errors and provides estimates of unmeasured states and forcing functions. The first of the algorithms is readily compared to Kalman-filter formulations that are currently used for state and parameter estimation. The second algorithm, consisting of a backward filter and forward smoother, has been implemented at Ames Research Center for use in flight-test analysis and to assist the National Transportation Safety Board in its investigation of aircraft accidents. Examples illustrating its application to a flight-test data consistency analysis and an aircraft motion analysis following an accident have been given. The good results obtained so far should direct attention to an apparently neglected but very useful tool for state and parameter estimation. It is now possible to determine whether our implementation will prove more effective than existing extended Kalman filter-smoother procedures. This study is part of a continuing research program.

Appendix A: Forward Filter—Backward Smoother

1) With x_0 and $[w(i)]$ obtained from the preceding iteration (or an initial guess), compute a nominal trajectory using Eq. (1), evaluate the residual sequence $[v(i)]$ using Eq. (2) and the performance measure using Eq. (3).

2) Solve the forward covariance filter, consisting of a time update:

$$\delta\bar{x}(i) = f_x \delta\hat{x}(i-1) - f_w w(i-1) \qquad \delta\hat{x}(0) = \bar{x}_0 - x_0$$

$$M(i) = f_x P(i-1) f_x^T + f_w Q f_w^T \qquad P(0) = P_0$$

and a measurement update:

$$\delta\hat{x}(i) = \delta\bar{x}(i) + K(i)e(i)$$

$$P(i) = [I - K(i)h_x]M(i)$$

where

$$e(i) = v(i) - h_x \delta\bar{x}(i) \qquad K(i) = M(i)h_x^T \bar{R}^{-1}$$

$$\bar{R} = R + h_x M(i) h_x^T$$

3) Solve the backward smoother:

$$\beta(i) = [I - K(i)h_x]^T [\lambda(i) - h_x^T R^{-1} e(i)]$$

$$\lambda(i-1) = f_x^T \beta(i) \qquad \lambda(N) = 0$$

During the backward pass, update the sequence $[w(i)]$ using

$$w(i-1) = -Q f_w^T \beta(i)$$

and determine an updated initial condition for the next iteration from

$$x_0 = \bar{x}_0 - P_0 \lambda_0$$

4) Iterate until changes in x_0, $[w(i)]$ are "sufficiently" small and the performance measure is minimized.

Appendix B: Backward Filter-Forward Smoother

1) With x_0 and $[w(i)]$ obtained from the preceding iteration (or an initial guess), compute a nominal trajectory using Eq. (1), evaluate the residual sequence $[v(i)]$ using Eq. (2) and the performance measure using Eq. (3).

2) Solve the backward information filter, consisting of a measurement update:

$$\beta(i) = \alpha(i) - h_x^T R^{-1} v(i) \qquad \alpha(N) = 0$$

$$S(i) = M(i) + h_x^T R^{-1} h_x \qquad M(N) = 0$$

and a time update:

$$\alpha(i-1) = f_x^T [\beta(i) - S(i) f_w \mathrm{d}(i)]$$

$$M(i-1) = f_x^T [I - f_w L(i)]^T S(i) f_x$$

where

$$\mathrm{d}(i) = \bar{Q}[f_w^T \beta(i) + Q^{-1} w(i-1)] \qquad L(i) = \bar{Q} f_w^T S(i)$$

$$\bar{Q} = [Q^{-1} + f_w^T S(i) f_w]^{-1}$$

3) Perform the Newton-Raphson computation:

$$\delta x_0 = -(P_0^{-1} + M_0)^{-1} [P_0^{-1}(x_0 - \bar{x}_0) + \alpha_0]$$

and solve the forward smoother:

$$\delta x(i+1) = f_x \delta x(i) + f_w \delta w(i) \qquad \delta x(0) = \delta x_0$$

with

$$\delta w(i) = -\mathrm{d}(i+1) - L(i+1) f_x \delta x(i)$$

4) Update x_0 and $[w(i)]$. Iterate until δx_0, $[\delta w(i)]$ are "sufficiently" small and the performance measure is minimized.

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